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Quantum Corrections to Noncommutative Solitons

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Noncommutative solitons are easier to find in a noncommutative field theory. Similarly, the one-loop quantum corrections to the mass of a noncommutative soliton are easier to compute, in a real scalar theory in $2+1$ dimensions. We carry out this computation in this paper. We also discuss the model with a double-well potential, and conjecture that there is a partial symmetry restoration in a vacuum state.

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1. Introduction

By now there is a fairly large literature on the subject of noncommutative solitons [1] in field theories as well as in string theory. However, there is little study on the quantum properties of these solitons. We will take first steps in this direction.

Noncommutativity in a sense drastically simplifies the task of the search for solitons. Noncommutativity is the cause of UV/IR connection, and drastically reduces the number of degrees of freedom. Technically, the field theory is essentially replaced by an one dimensional matrix model in the large noncommutativity limit. And the *essential* dynamic degrees of freedom are the eigen-values of the matrix. A soliton may be called a single eigen-value soliton. In the same vein, noncommutativity also simplifies the computation of quantum corrections to the spectrum of solitons. The reason is that in the large θ limit, one can integrate out the angular variables in the unitary matrix, in the decomposition $\phi = U\Lambda U^+$, the result is a modification of the wave functions by the Vandermonde determinant with the Hamiltonian remaining unchanged. To be more accurate, excitations of angular variables will cause quantum corrections suppressed by g^2, the coupling constant in question. Thus the problem of computing quantum corrections to the spectrum of solitons boils down to a simple quantum mechanical problem. This is to be contrasted to the problem of computing one-loop correction to a conventional soliton in a usual quantum field theory, where ingenious techniques are often required to carry out the calculations. In a noncommutative field theory, solving the differential equation is reduced to solving an algebraic equation, and computing a functional determinant is reduced to computing a number. We will carry out the "one-loop" computation in the next section. We will treat the spatial kinetic term as a perturbation, since it is suppressed by $1/\theta$.

The same reason prompts one to suspect that the folklore of symmetry breaking is no longer valid in a noncommutative field theory. We will conjecture in sect.3 that indeed in a $2+1$ dimensional noncommutative scalar field theory, symmetry is partially restored in a ground state in the large θ limit. We will provide some evidence to this effect. For finitely many eigenvalues, the Z_2 symmetry with a double-well potential is not broken. In the spacetime picture, this corresponds to the partial symmetry restoration in a finite region whose size depends on θ. This phase breaks translational invariance thus deserves further study.

2. One-loop corrections

We will mostly consider a noncommutative real scalar in $2+1$ dimensions. The action is

$$S=\frac{1}{2\pi g^2\theta}\int dtd^2x\left(\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2}(\partial_i\phi)^2-V(\phi)\right), \tag{2.1}$$

where all products are understood as the star product, $V(\phi)$ is a polynomial in ϕ, and is assumed to be bounded from below. The coupling constant $g^2\theta$ is chosen to contain a factor θ, the noncommutative parameter appearing in

$$[x_1,x_2]=i\theta. \tag{2.2}$$

This choice is purely for later convenience. We define the coupling constant in such a way that the leading power in $V(\phi)$ has a dimensionless numerical coefficient. For instance, in the case of a ϕ^4 theory, $V(\phi)=1/4\phi^4+\ldots$. The dimension of $g^2\theta$ can be determined as follows. First, from the kinetic terms, one finds

$$[g^2\theta]=[\phi^2]L.$$

If the leading power of $V(\phi)$ is n, then by comparing this leading term with the kinetic term, one finds

$$[\phi]=L^{-\frac{2}{n-2}}.$$

Of course in order to have an interacting theory and solitons, $n>2$. Since the dimension of θ is L^2, finally we have

$$[g^2]=L^{-\frac{n+2}{n-2}}. \tag{2.3}$$

For $n=4$, $[g^2]=M^3$, and for $n=6$, $[g^2]=M^2$. Without noncommutativity, the former theory is super renormalizable, and th latter is marginally renormalizable.

To discuss the GMS solitons, it is convenient to rescale the spatial coordinates $x_i\to\sqrt{\theta}x_i$ Furthermore, it is useful to replace the integration over x_i by the trace using $\int d^2x=2\pi\mathrm{tr}$ after the rescaling. The Hamiltonian is written in a neat form

$$H=\frac{1}{g^2}\mathrm{tr}\left(\frac{1}{2}(\partial_t\phi)^2-\frac{1}{2\theta}[x_i,\phi]^2+V(\phi)\right). \tag{2.4}$$

In the large θ limit, we see that we can drop the spatial derivative terms $[x_i,\phi]^2$. The above Hamiltonian describes a matrix model, one that is a slight modification of the old

familiar $c=1$ matrix model. To cast the Hamiltonian in the more familiar matrix form, introduce the creation and the annihilation operators

$$a = \frac{1}{\sqrt{2}}(x_1 + ix_2), \quad a^+ = \frac{1}{\sqrt{2}}(x_1 - ix_2) \tag{2.5}$$

satisfying $[a, a^+] = 1$. Field ϕ is operator valued, thus can be written in the form

$$\phi = \sum_{m,n} \phi_{mn} |m\rangle\langle n|, \tag{2.6}$$

where $|m\rangle$ are the normalized eigen-states of the number operator a^+a. Apparently, the multiplication operation becomes that of matrices. The Hamiltonian (2.4) does not have a $U(\infty)$ gauge symmetry. However, if one throws away the spatial kinetic term, which is small in the large θ limit, the model possesses a global $U(\infty)$ symmetry.

In the large θ limit, one can always diagonalize ϕ:

$$\phi = \sum_m \lambda_m |m\rangle\langle m|, \tag{2.7}$$

and the Hamiltonian becomes simply a sum of infinitely many decoupled terms

$$H_0 = \frac{1}{g^2} \sum_m \left(\frac{1}{2} (\partial_t \lambda_m)^2 + V(\lambda_m) \right). \tag{2.8}$$

For a static configuration, to minimize the energy, we need to minimize all the individual terms, so that

$$\frac{dV(\lambda_m)}{d\lambda_m} = 0. \tag{2.9}$$

Thus the static dynamics is dictated by the local minima of the potential V. We shall always assume the global minimum of V be 0 in this section, so the vacuum state has a zero energy classically. GMS observed that, if there exists another local minimum $\bar{\lambda}$ with $V(\bar{\lambda})$, then there exist solitons $\phi = \bar{\lambda}|m\rangle\langle m|$ and multi-solutions as suppositions of these solitons. Although it appears that these results are rather trivial from the matrix model perspective, the solitons are nontrivial as they are truly lumps in the two dimensional space. Translated into function of x_i, $|m\rangle\langle m|$ is a polynomial with a Gaussian damping factor.

It is the special feature of a noncommutative field theory that the supposition of all single eigen-value solitons give the false vacuum, since

$$\phi = \sum \bar{\lambda} |m\rangle\langle m| = \bar{\lambda}, \tag{2.10}$$

namely, although each soliton is well-localized around $x_i = 0$, their supposition gives rise to a constant configuration of the scalar field.

For a single soliton, the classical energy is degenerate regardless which eigen-value λ_m is excited to $\bar{\lambda}$:

$$E_m = \frac{1}{g^2} V(\bar{\lambda}). \tag{2.11}$$

This degeneracy is lifted by turning on the spatial kinetic energy. For the diagonal configuration (2.7), this term $-(1/g^2\theta)\mathrm{tr}[a, \phi][a^+, \phi]$ reads

$$\frac{1}{g^2\theta} \sum_m m(\lambda_m - \lambda_{m-1})^2. \tag{2.12}$$

Consider the more general situation when all λ_m are dynamic, namely when they depend on time, the Hamiltonian is

$$H = \frac{1}{g^2} \sum_m \left(\frac{1}{2}(\partial_t \lambda_m)^2 + \frac{1}{\theta} m(\lambda_m - \lambda_{m-1})^2 + V(\lambda_m) \right). \tag{2.13}$$

Assume the global minimum of V is at $\lambda = 0$, then to the first order in $1/\theta$, the m-th soliton has an energy

$$E_m = \frac{1}{g^2} \left(V(\bar{\lambda}) + \frac{2m+1}{\theta} \bar{\lambda}^2 \right). \tag{2.14}$$

The degeneracy is lifted by a small term suppressed by $1/\theta$. We computed the above correction by simply substituting the unmodified soliton into (2.13). The exact static solution when the spatial kinetic energy is turned on is different from the simple ansatz $\lambda_m = \bar{\lambda}$, $\lambda_n = 0$, $n \neq m$. However, this modification does not change the result (2.14). To see this, we can expand the Hamiltonian (2.13) around the unmodified soliton configuration and find that the correction to ϕ is suppressed by $1/\theta$, all terms in the expansion except for the one showing up in (2.14) are then suppressed by $1/\theta^2$ upon substitution of the modified solution.

To discuss the quantum corrections to the spectrum of solitons, again we will ignore the spatial kinetic term first. Without the presence of this term, it is well-known in the old matrix models that one can integrate out angular variables first. Decompose the full matrix as written in (2.6) into $\phi = U \Lambda U^+$, where Λ is the full diagonal form as in (2.7), and U is a unitary operator $U \in U(\infty)$. To see how integrating out U affects the resulting theory, it is better to work with the path integral:

$$\int [d\Lambda][dU] \exp(iS). \tag{2.15}$$

Now if S contains only the time kinetic term and the potential term, U can be explicitly integrated out [2,3]. Consider the propagator between states with fixed initial Λ_i and fixed final Λ_f, the path integral yields

$$\int [d\Lambda]\Delta(\Lambda_i)\Delta^{-1}(\Lambda_f)\exp(iS(\Lambda)), \tag{2.16}$$

where $S(\Lambda)$ is the action by substituting $\phi = \Lambda$ into $S(\phi)$, and $\Delta(\Lambda)$ is the Vandermonde determinant

$$\Delta(\Lambda) = \prod_{m<n} (\lambda_m - \lambda_n). \tag{2.17}$$

Note that in (2.16) we integrated out U_i but not U_f, otherwise we would get a factor $\Delta(\Lambda_f)$ instead of $\Delta^{-1}(\Lambda_f)$. Result (2.16) can be obtained by put the matrix model on a discrete time lattice, integration of angular variables yields many Vandermonde determinants, and all the intermediate determinants cancel. We thus see that integrating out the angular variable U merely modifies the wave function $\Psi(\Lambda) \to \Delta(\Lambda)\Psi(\Lambda)$ without changing the Hamiltonian (2.8). This antisymmetric factors comes from the phase space of the angular variables.

When the initial wave function is a nontrivial function of these variables, integrating out them is actually more subtle than described above. In any case, if one uses the arguments in the first reference in [3], one will see that excitations of angular variables bring in corrections only at the order g^2, one-loop higher than what we will be interested in this paper. All these angular variables are flat directions. In particular, the mode corresponding to the center of the soliton is a collection of these variables (the U matrix is simply $\exp(i(\alpha a + \bar{\alpha} a^+))$). Since the mass of the soliton is proportional to g^{-2}, the kinetic term of the soliton is roughly $(g^2/\lambda^2)p_\alpha^2$, we see that indeed this term also brings in correction at the order g^2, as it should be the case in general [1]. By freezing angular directions except taking the phase space into account, we also demand the soliton stay at rest.

It becomes important to remember that there is a spatial kinetic term in the full action breaking the $U(\infty)$ symmetry. Without this term, we would start with a totally symmetric wave function $\Psi(\Lambda)$ and end up with a totally asymmetric wave function $\Delta(\Lambda)\Psi(\Lambda)$, and

[1] The problem of treating the collective coordinates is always subtle in a commutative field theory, we see that in the noncommutative case, it is relatively simple, although the center degree of freedom α is composed of infinitely many angles and requires a careful treatment.

each eigen-value λ_m is transmutated into a fermion, as in the old matrix models. In this case the vacuum cannot be $\lambda_m = 0$, and there are no soliton solutions at all! Thus, even though the classical perturbation introduced by the spatial kinetic term is suppressed by $1/\theta$ as in (2.13), its quantum correction is enormous to change the vacuum structure.

We will first ignore the spatial kinetic term in discussing the quantum corrections to the spectrum of solitons, and consider its effects later. Since there is no restriction on the wave function, all eigen-values are completely decoupled, as the Hamiltonian (2.8) clearly indicates. The canonical momentum of λ_m computed from (2.8) is

$$p_m = \frac{1}{g^2}\partial_t \lambda_m. \tag{2.18}$$

With this identification, the Hamiltonian becomes

$$H = \sum_m \left(\frac{g^2}{2} p_m^2 + \frac{1}{g^2} V(\lambda_m) \right). \tag{2.19}$$

Clearly, if $\lambda_m = 0$ is the absolute minimum point of $V(\lambda_m)$, then the ground state $\Psi_0(\lambda_m)$ is localized around this minimum, and the ground state of the whole system is given by $\Psi_0(\Lambda) = \prod_m \psi_0(\lambda_m)$. The classical soliton with a $\lambda_m = \bar{\lambda}$ is unstable against quantum tunneling. We will not be interested in this tunneling in this article. What we are interested in is the perturbative quantum corrections to the classical energy of the soliton, thus the full perturbative energy is a Taylor series in g^2. While the quantum tunneling is suppressed by a factor $\exp(-c/g^2)$ with the constant c depending on the details of the potential V.

The first order correction to the energy is of order $O(g^0)$ and is normally called the one-loop correction. While it is often a quite technical problem to compute this one-loop correction in a usual quantum field theory, the computation is almost trivial in the noncommutative field theory without $U(\infty)$ symmetry. We need only expand $V(\lambda_m)$ around $\bar{\lambda}$ to the second order in $\Delta\lambda_m = \lambda_m - \bar{\lambda}$. The Hamiltonian for λ_m to this order reads

$$H_m = \frac{1}{g^2} V(\bar{\lambda}) + \frac{g^2}{2} p_m^2 + \frac{1}{2g^2} V''(\bar{\lambda})(\Delta\lambda_m)^2, \tag{2.20}$$

where $V''(\bar{\lambda})$ is the second derivative of V at $\bar{\lambda}$ and is positive. Thus the wave function for λ_m is that of the ground state of a harmonic oscillator, it is just

$$\Psi(\Delta\lambda_m) = \exp(-\frac{\sqrt{V''(\bar{\lambda})}}{2g^2}(\Delta\lambda_m)^2). \tag{2.21}$$

More generally, the n-th excited state has an energy

$$E = \frac{1}{g^2} V(\bar{\lambda}) + (n + \frac{1}{2}) \sqrt{V''(\bar{\lambda})}. \tag{2.22}$$

We might think this is the quantum corrected energy of the soliton with n quanta bound to it. This is not true, we need to take the energy of vacuum into account. When all eigen-values stay in their ground state, the classical energy vanishes. Quantum mechanically, to the first order in g^2, there is a correction

$$\frac{1}{2} \sqrt{V''(0)} \tag{2.23}$$

to the energy of a single eigen-value. The sum of these corrections diverges as usual in a field theory. For a soliton excited to its n-th level, all other eigen-values still stay in their ground state around $\lambda = 0$ except λ_m, thus the subtracted energy of the soliton is

$$E_m = \frac{1}{g^2} V(\bar{\lambda}) + \frac{1}{2} (\sqrt{V''(\bar{\lambda})} - \sqrt{V''(0)}) + n \sqrt{V''(\bar{\lambda})}. \tag{2.24}$$

The above is the true quantum corrected energy of a soliton bound to n quanta. When $n = 0$, we have the one-loop corrected energy of the soliton. The condition for the one-loop correction to be much smaller than the classical result is

$$g^2 \ll \frac{V(\bar{\lambda})}{|\sqrt{V''(\bar{\lambda})} - \sqrt{V''(0)}|}. \tag{2.25}$$

Next we consider the correction brought about by the spatial kinetic term. When this term is present, it is impossible to integrate out the angular variables U as before. However, since this term is suppressed by $1/\theta$, it is reasonable to assume that integrating out U will result a correction to the wave function by $\Delta(\Lambda)$ and a small correction to the Hamiltonian. We will ignore this correction to the Hamiltonian. We have seen in (2.14) that the classical correction induced by the spatial kinetic term is suppressed by a factor $1/\theta$, and not surprisingly, we will see that the quantum correction induced by this term is further suppressed by a factor g^2. Denote $H = H_0 + \Delta H$, where H_0 is the Hamiltonian as in (2.8), and ΔH is the spatial kinetic term. Let Ψ_i be the eigenstates of H_0, using the standard perturbation theory, the eigen-values of H is determined by the following equation

$$\det[\langle \Psi_i | \Delta H | \Psi_j \rangle - \delta_{ij}(E - E_i)] = 0. \tag{2.26}$$

Consider Ψ_i to be the wave functions when all λ_n $n \neq m$ are excited around $\lambda = 0$, and the m-th eigen-value λ_m is excited around $\bar{\lambda}$. Since the energy-levels E_i are not degenerate, it is rather easy to execute the perturbation calculation. We first assume that the off-diagonal elements $\langle \Psi_i | \Delta H | \Psi_j \rangle$ can be ignored, then the correction to E_m is simply

$$\langle \Psi_m | \Delta H | \Psi_m \rangle = \frac{1}{g^2 \theta} \sum_n (2n+1) \langle \Psi_m | \lambda_n^2 | \Psi_m \rangle, \tag{2.27}$$

where Ψ_m is given by the product of (2.21) and the ground state wave functions of λ_n centering around $\lambda_n = 0$. In getting the above expressing we have observed that the expectation value of the nearest-neighbor coupling $\lambda_n \lambda_{n+1}$ vanishes. (2.27) is easy to compute, we have

$$\frac{1}{\theta}(m + \frac{1}{2}) \frac{1}{\sqrt{V''(\bar{\lambda})}} + \frac{1}{\theta} \sum_n{}' (n + \frac{1}{2}) \frac{1}{\sqrt{V''(0)}}, \tag{2.28}$$

where the primed sum does not include $n = m$. The above sum is divergent. To get a finite result, again we need to subtract the correction to the vacuum energy, thus

$$\langle \Psi_m | \Delta H | \Psi_m \rangle = \frac{1}{\theta}(m + \frac{1}{2}) (\frac{1}{\sqrt{V''(\bar{\lambda})}} - \frac{1}{\sqrt{V''(0)}}). \tag{2.29}$$

Compared with the classical correction in (2.14) we see that indeed this term is further suppressed by a factor g^2.

Next we shall argue that the off-diagonal elements of ΔH can be ignored. To see this, we use another method to get the same result as in (2.29). The spatial kinetic term can be divided into two parts. One part is a sum of the terms

$$\frac{1}{g^2 \theta}(2n+1)\lambda_n^2.$$

The second part is a sum of the nearest neighbor coupling

$$-\frac{1}{g^2 \theta} n \lambda_n \lambda_{n-1}.$$

Now treat the second part as a perturbation, while the first part is included in H_0, we see that the total one-loop quantum correction is

$$\frac{1}{2}\sqrt{V''(\bar{\lambda}) + 2(2m+1)/\theta}. \tag{2.30}$$

Expanding the above result we get the second term to be the one in (2.29). The higher order terms are suppressed by more factors of $1/\theta$. Further, it is easy to see that in the

nearest neighbor coupling, the elements of λ_n between the ground state and the excited states are all vanishing except the first excited state. According to the perturbation theory, the correction induced by this is proportional to

$$(\langle\Psi|1/(g^2\theta)\lambda_n\lambda_{n-1}|\Phi\rangle)^2/(E_1-E_0),$$

where E_1-E_0 is the energy difference between state Φ and state Ψ. Due to the appearance of square of the off-diagonal element, this term is suppressed by $1/\theta^2$. The off-diagonal element is independent of g.

To summarize, the corrected energy of the m-th single soliton is

$$\begin{aligned} E_m = &\frac{1}{g^2}(V(\bar{\lambda})+\frac{2m+1}{\theta}\bar{\lambda}^2)+\frac{1}{2}(\sqrt{V''(\bar{\lambda})}-\sqrt{V''(0)}) \\ &+\frac{1}{\theta}(m+\frac{1}{2})(\frac{1}{\sqrt{V''(\bar{\lambda})}}-\frac{1}{\sqrt{V''(0)}}). \end{aligned} \tag{2.31}$$

If we demand the quantum correction induced by the spatial kinetic term be much smaller than its classical counterpart, we must impose

$$g^2 \ll \frac{\bar{\lambda}^2\sqrt{V''(\bar{\lambda})V''(0)}}{|\sqrt{V''(\bar{\lambda})}-\sqrt{V''(0)}|}. \tag{2.32}$$

Further, if we demand the quantum correction induced by the spatial kinetic term be much smaller than the one in the decoupled system, there must be

$$\frac{1}{\theta} \ll \frac{\sqrt{V''(\bar{\lambda})V''(0)}}{m+\frac{1}{2}}. \tag{2.33}$$

This condition may be violated for a sufficiently large m.

To have a feeling about the result (2.31), consider a ϕ^4 theory. $V'(\lambda)$ is a polynomial of degree 3, so it has three zeros. To have a soliton, there must be two zeros at $\lambda=0$ and $\lambda=\bar{\lambda}$. The third zero must be real too, denote it by μ. Thus

$$V'(\lambda)=\lambda(\lambda-\bar{\lambda})(\lambda-\mu). \tag{2.34}$$

The relevant data for quantum corrections are

$$V''(0)=\mu\bar{\lambda}, \quad V''(\bar{\lambda})=\bar{\lambda}(\bar{\lambda}-\mu). \tag{2.35}$$

For both of them to be positive, μ and $\bar{\lambda}$ must have the same sign, choose them to be positive; and $\bar{\lambda} > \mu$. $\lambda = \mu$ is a local maximum of V. The potential is a deformed double-well. Integrating (2.34) to get V and set $V(0) = 0$, we find

$$V(\bar{\lambda}) = \frac{1}{12}\bar{\lambda}^3(2\mu - \bar{\lambda}). \tag{2.36}$$

This value must be positive, therefore $2\mu > \bar{\lambda}$. Applying this condition to (2.35), we have $V''(0) > V''(\bar{\lambda})$. We therefore see that the leading quantum correction in (2.31) is negative, the quantum correction due to the kinetic term is positive.

Quantum corrections to multiple solitons sitting at the same point can be readily carried out. In this case a few λ's are excited to $\bar{\lambda}$ and there is no interaction at the classical level, since even the spatial kinetic term vanishes. Quantum mechanically, there is no correction at the one loop level to the interaction energy, since the cross terms such as $\Delta\lambda_m \Delta\lambda_{m-1}$ has vanishing expectation value.

It is also interesting to study quantum correction to the energy of two separated solitons. The simplest case is a soliton localized at $x = 0$, given by $\bar{\lambda}|0\rangle\langle 0|$, and a soliton localized at a another point given by $\bar{\lambda}|z\rangle\langle z|$. The second soliton can not be written as a diagonal matrix, so new technique is required.

3. Partial Symmetry Restoration

In this section we will consider the noncommutative ϕ^4 theory in $2+1$ dimensions. We assume the potential take the form

$$V(\phi) = \frac{1}{4}\phi^4 - \frac{\mu^2}{2}\phi^2. \tag{3.1}$$

According to the dimensional analysis performed in the previous section, ϕ has an energy dimension, so does μ, and $[g^2] = M^3$. Classically, as in the ordinary commutative field theory, the vacua are degenerate at $\phi = \pm\mu$. There is no soliton unlike what was studied in the previous section. Quantum mechanically, the vacua remain degenerate for the commutative field theory. What happens in the noncommutative case?

We conjecture that in a true ground state, the vacuum expectation values of the first few λ_m are zero, with the largest m determined roughly by $\mu^2\theta$. The remaining infinitely many eigen-values stay in the valley of the double-well potential. Such a vacuum seems to break translational invariance, thus it is something like a "stripe phase". We make this

conjecture for the zero temperature theory, unlike what was considered recently by Gubser and Sondhi [4], where an Euclidean noncommutative field theory was studied.

It is well-known that for a quantum mechanical system with the double-well potential (3.1), the degeneracy of two perturbative ground states centering around $\phi = \pm\mu$ is lifted nonperturbatively, due to the tunneling effects. Let the Hamiltonian of this quantum mechanics be

$$H = \frac{1}{g^2}(\frac{1}{2}(\partial_t\phi)^2 + V(\phi)), \tag{3.2}$$

with $V(\phi)$ given in (3.1). Let $E_0(g^2)$ be the ground state energy computed by the perturbative expansion method. The nonperturbative effects lift the degeneracy and the two eigen-values of the Hamiltonian become $E_\pm(g^2)$, the difference between these two energies can be computed by the WKB method and turns out to be of the form [5]

$$E_+(g^2) - E_-(g^2) = c_1\frac{\mu^{5/2}}{g}e^{-\frac{c_2\mu^3}{g^2}}, \tag{3.3}$$

where $c_{1,2}$ are positive dimensionless constants . This result is good in the limit $g^2/\mu^3 \ll 1$. Note that the true ground state energy E_- is lower than the perturbative ground state energy E_0.

In a quantum field theory with potential (3.1), there are two extremal limits. In one extreme, the spatial derivative terms can be ignored, and in this case the field theory is ultra-local. At every spatial point, ϕ fluctuates independently. If we divide space into cells with a small volume v, then the effective coupling is g^2/v, very large in the UV limit. In this limit the ϕ^4 term dominates the dynamics, and the degeneracy is surely lifted in this limit, although the result (3.3) can no longer be trusted. In the other extremal limit, the spatial kinetic term is important, so ϕ is forced to fluctuate collectively. Denote this zero mode by ϕ_0, its effective coupling is g^2/V where V to the infrared cut-off on the whole volume. Since in the large volume limit, the effective coupling is small, and the energy gap (3.3) becomes accurate in this limit:

$$E_+ - E_- = c_1\frac{\sqrt{V}}{g}e^{-\frac{c_2V}{g^2}}. \tag{3.4}$$

In the thermodynamic limit $V = \infty$, the energy gap tends to 0 rapidly, the degeneracy is resumed, the Z_2 symmetry is spontaneously broken.

In the realistic case, one has to study the system carefully, taking into account the spatial kinetic energy. It turns out for a real scalar field, symmetry is always broken in

2 dimensions and above. For a complex scalar, the kinetic term does not suppress the infrared correlations of the Goldstone boson in 2 dimensions, and there is no symmetry breaking.

We now turn to our $2+1$ dimensional noncommutative ϕ^4 theory. As emphasized in the previous section, it is a difficult problem to integrate out the off-diagonal modes of ϕ when the spatial kinetic term is present. Nevertheless we assume that the correction induced by this term is smaller than its classical value when evaluated for $\phi = \Lambda$, a diagonal matrix. Thus the net result of integrating out U in the decomposition $\phi = U\Lambda U^+$ is the Vandermonde determinant modifying the wave function. The Hamiltonian is still given by (2.13). For convenience, we write down this Hamiltonian again

$$H = \frac{1}{g^2}\sum_m \left(\frac{1}{2}(\partial_t\lambda_m)^2 + \frac{1}{\theta}m(\lambda_m - \lambda_{m-1})^2 + \frac{1}{4}\lambda_m^4 - \frac{1}{2}\mu^2\lambda_m^2\right). \qquad (3.5)$$

In the limit $\theta = \infty$, the spatial kinetic term drops out, the system becomes of a collection of ϕ^4 quantum mechanical systems, and the vacuum state is the state when all eigen-values λ_m stay in their ground state of energy E_-, so there is no spontaneous symmetry breaking. The first excited states are degenerate: When all λ_m are in their ground state except only one eigen-value in the state with energy E_+. The energy of these first excited state, after subtracted the vacuum energy, is given by $E_+ - E_-$ as in (3.3), and curiously, is very small when $g^2/\mu^3 \ll 1$. We conclude that in the large θ limit, there is no symmetry breaking, and there is an energy gap which is nonperturbative in nature. The fact that a $2+1$ dimensional noncommutative scalar field theory does not exhibit spontaneous symmetry breaking is due to the drastic reduction of effective number of degrees of freedom. In a way, it is similar to the ultra-local commutative field theory. It is also different from the latter, since the effective coupling constant is always g^2 and all the eigen-values λ_m have the same amount of fluctuations, while in the ultra-local field theory, the fluctuations become violent in the UV limit, since the effective coupling increases in the UV limit.

When the kinetic term is turned on, the tremendous degeneracy of the first excited states is lifted. Hamiltonian (3.5) describes a chain on a half infinite line with nearest neighbor coupling. When θ large and m small, the spatial kinetic term is not strong, one would imagine these eigen-values essentially behave like decoupled eigen-values, and the Z_2 symmetry is not broken for them. For m large enough, the spatial kinetic term starts to play an role, we will argue that it will derive almost all eigen-values to their minimal point.

The one dimensional chain (3.5), although looks simple, can not be solved exactly. We will use perturbation arguments. Classically, since the kinetic term is positive definite, the minimal energy is achieved when $\lambda_m = \pm\mu$. Quantum mechanically, all λ_m fluctuate, it is a tricky question as to what we shall take as our unperturbed Hamiltonian. We shall consider two situations separately. In one hypothetical case, Z_2 symmetry is not broken for all λ's. In the other case, Z_2 symmetry is broken.

Consider the assumption that the symmetry is not broken for all eigenvalues. In this case we will not take the Hamiltonian in (2.8) as the unperturbed one, rather, we take

$$H_0 = \frac{1}{g^2}\sum_m \left(\frac{1}{2}(\partial_t\lambda_m)^2 + (-\frac{1}{2}\mu^2 + \frac{2m+1}{\theta})\lambda_m^2 + \frac{1}{4}\lambda_m^4\right) \tag{3.6}$$

as the unperturbed Hamiltonian. The perturbation is then the nearest neighbor coupling

$$H_I = -\frac{2}{g^2\theta}\sum_m m\lambda_m\lambda_{m-1}. \tag{3.7}$$

Without perturbation (3.7), the system described by (3.6) is simple. The effective mass square for λ_m is

$$\mu_m^2 = \frac{2(2m+1)}{\theta} - \mu^2.$$

When this parameter is positive, the potential has a minimum at $\lambda_m = 0$. In the following argument, we ignore the finitely many eigen-values for which μ_m^2 is negative, since contribution of these eigenvalues to energy is always finite. Now the zero-point energy for λ_m with positive μ_m is simply $(1/2)\mu_m$. Since all the wave functions of λ's are even functions, the perturbation (3.7) has vanishing expectation value in this "supposed to be" vacuum, the vacuum energy is given by, ignoring higher loop corrections

$$E_0 = \sum_{m=M}\frac{\mu_m}{2}, \tag{3.8}$$

where M is the smallest m making μ_m^2 positive. If we introduce a cut-off $m = N$ on the sum, then the second sum scales roughly as

$$\frac{2}{3\sqrt{\theta}}N^{3/2}. \tag{3.9}$$

As we remarked before, if we do not take the nearest neighbor coupling as a perturbation, then there are two classical minima when all $\lambda_m = \mu$ or all $\lambda = -\mu$. We now study the quantum fluctuation around one of them, say $\lambda = \mu$. Each eigenvalue contributes to

the classical energy an amount$-\mu^4/(4g^2)$, and each contribute a zero point energy $\mu/\sqrt{2}$. The sum of these contribution diverges as

$$-(\frac{\mu^4}{4g^2}-\frac{\mu}{\sqrt{2}})N. \tag{3.10}$$

We are assuming the perturbation theory is correct, so $g^2/\mu^3 \ll 1$, the above contribution is negative. Expanding the whole Hamiltonian around $\lambda_m=\mu$, and denote $\Delta\lambda_m=\lambda_m-\mu$, the Hamiltonian, for $m\geq M$, is given by a Gaussian part

$$H_0=\frac{1}{g^2}\sum_m\left(\frac{1}{2}(\partial_t\Delta\lambda_m)^2+(\mu^2+\frac{2m+1}{\theta})\Delta\lambda_m^2\right), \tag{3.11}$$

a perturbation

$$H_I=-\frac{2}{g^2\theta}\sum_m m\Delta\lambda_m\Delta\lambda_{m-1}, \tag{3.12}$$

and a higher order sum. The zero-point fluctuation determined by (3.11) is

$$\sum_m\frac{1}{2}\left(\frac{2(2m+1)}{\theta}+2\mu^2\right)^{1/2}. \tag{3.13}$$

This term also diverges as the sum (3.8) in the same fashion as in (3.9). But the difference between this sum and (3.8) is roughly

$$\frac{3}{4}\mu^2\sqrt{\theta}\sqrt{N}. \tag{3.14}$$

Although it is positive, in the limit $N\to\infty$, this term is overwhelmed by the negative energy (3.10), so the symmetry broken phase $\lambda=\mu$ has much smaller energy.

For λ_m with μ_m^2 negative, the story can be completely different. For these eigen-values, even in the symmetric phase, the energy is negative, and if we adopt (3.7) as perturbation, it has zero expectation value in the symmetric state. For this term to be a small perturbation, we require its element between the symmetric state with energy E_- and the first excited state with energy E_+ to be smaller than $(\Delta E\mu_m)^{1/2}$. We find the condition

$$\frac{m^2}{\theta^2}\ll c_1g^3|\mu_m|^{-1/2}\exp(-\frac{c_2|\mu_m|^3}{g^2}), \tag{3.15}$$

it can be satisfied for small m and large θ.

Thus, it is likely that the one-dimension chain described by (3.5) has a zero-temperature phase in which finitely many eigen-values are in the symmetric phase. These

eigen-values still have an effective negative μ_m^2, thus m is roughly smaller than $\mu^2\theta$. As pointed out in [1], the corresponding state $|m\rangle\langle m|$ associated with this eigen-value has a spatial size $\sqrt{m}$, or in terms of the original, un-rescaled coordinates, its size is $\sqrt{m}\sqrt{\theta}$. With the condition $m < \mu^2\theta$, we find the size be smaller than $\mu\theta$. A more careful analysis should incorporate a condition such as (3.15). For instance, one may require that the spatial kinetic energy between the two neighboring eigen-values to be smaller than the lowering of energy due to quantum tunneling, given in (3.3). The former is roughly $m\mu^2/(g^2\theta)$, for the difference $\lambda_m - \lambda_{m-1}$ is roughly μ in a symmetric phase. Thus,

$$m < g\sqrt{\mu}e^{-\frac{c_2\mu^3}{g^2}}\theta, \tag{3.16}$$

so the physical size of symmetry restoration

$$\sqrt{m\theta} \sim g^{1/2}\mu^{1/4}e^{-\frac{c_2\mu^3}{2g^2}}\theta = f(g,\mu)\theta. \tag{3.17}$$

This result may just reflect the fact that in noncommutative space, the fluctuation of the field within size $f(g,\mu)\theta$ is unavoidably large due to space-space uncertainty. This size grows when θ is increased. There ought to be a close relation between the partial symmetry restoration and the UV/IR connection at the perturbative level found in [6].

It goes without saying that the above result is to be taken only as a substantiated conjecture, since we have ignored dynamic fluctuations of angular variables, and their effect introduced through the spatial kinetic term.

The partial symmetry breaking is interesting. However it raises a puzzle: In such a phase, translational invariance seems to be broken. In a sense this vacuum is similar to the stripe phase discussed in [4]. The simple reason that the spontaneously translational symmetry-breaking is possible is that the translation generators are part of the angular variables, which are all flat directions and can spontaneously stay at a point. Further detailed study is desirable to clarify how spontaneous translational symmetry-breaking occurs. Nevertheless its occurrence is not unique. As discussed in [16,17], tachyon condensation in a D-brane anti-D-brane system also spontaneously breaks translational invariance in matrix models, the condensation occurs in a finite region at the classical level. For other discussions of the noncommutative tachyon, see [25]. The cause of this phenomenon is the opposite sign of the B field on the two D-branes. In the system discussed here, the "tachyon" field does not condense in a finite region.

4. Conclusion and Discussions

We have in this paper mainly considered a noncommutative field theory with a real scalar, in $2+1$ dimensions. This dimensionality is interesting in string theory, for instance in a brane-anti-brane system, lower dimensional branes are constructed as solitons of co-dimension 2 [7-18]. However, to get results applicable to string theory, the gauge field must be introduced. Results obtained here will be modified with the presence of the gauge field. The most important modification will be that the complete symmetry breaking will become possible again, for otherwise Sen's tachyon condensation scenario would be wrong, this is unlikely to happen. It should also be interesting to study quantum corrections to the exact solitons discussed recently in [19-20] and other solitons involving the gauge field [21].

When there is a gauge field in the system, unlike the commutative case, a scalar can remain real and is still coupled to the gauge field in the adjoint representation. A tachyon on a unstable D-brane in a type II string theory realizes this situation. The problem of computing quantum corrections to a unstable soliton in a gauge theory is complicated by two things in a noncommutative gauged system. First, even though one can still diagonalize a real scalar in the operator representation, one is left with the problem of treating the time component of the gauge field. Second, the spatial derivative terms and the Yang-Mills term are important. To see this, recall that a covariant derivative $D_\mu\phi$ is replaced by a commutator $[X_\mu, \phi]$ in the matrix representation [22] [23]. Rescaling of coordinates $x_i \to \sqrt{\theta}x_i$ is absorbed into a rescaling of the gauge field, so the Yang-Mills term $\mathrm{tr}[X_\mu, X_\mu]^2$ also receives a rescaling, the effective Yang-Mills coupling is $g^2\theta$ rather than g^2. Thus the Yang-Mills part is strongly coupled compared to the self coupling of the scalar. In a ϕ^4 theory, the effective dimensionless coupling is $g^2\theta/\mu$.

To discuss gauge symmetry breaking, one must first understand better the Higgs mechanism in this setting [24]. We expect no surprises here.

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